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PLATO'S THEORY OF NUMBER

In a well-known passage Aristotle ascribes to Plato, or as some think to his followers, the dictum, *ὁ γὰρ ἀριθμός ἐστιν ἐκ ἑνὸς καὶ τῆς δυάδος τῆς ἀορίστου* (*Met.* 1081a, 14), 'Number is (formed) from the one (unit, monad) and the undetermined (indefinite, unbounded) dyad (duality)', but what this apparently simple statement means has remained a mystery until modern times. In other passages Aristotle expands it to explain that the indefinite duality is a duality of the great and small, e.g., *ὡς μὲν οὖν ὕλην τὸ μέγα καὶ τὸ μικρὸν εἶναι ἀρχάς, ὡς δ' οὐσίαν τὸ ἕν. ἐξ ἐκείνων γὰρ κατὰ μέθεξιν τοῦ ἑνὸς εἶναι τοὺς ἀριθμούς* (*Met.* 987b20–22). 'As the matter (of number) he posits the great and small for principles, as substance the one; for by the mixture of the one with them he says numbers (arise).' This identification of the dyad with the great and small, elsewhere called *τὸ ἄνισον* ('the unequal') and *τὸ ἄπειρον* ('the unbounded'), gives a first clue to its nature. In a notable article in *Mind* 35 (1926), 419–40, continued in vol. 36, (1927), 12–33, and amplified by D'Arcy Wentworth Thompson in vol. 38 (1929), 43–55, A. E. Taylor first suggested a connexion between the indefinite duality and the modern theory of continued fractions. In the light of subsequent research in the history of Greek mathematics it may now be asserted with a high degree of confidence that his conjecture was almost certainly correct; but it was then no more than a conjecture, and when he looked for confirmation he looked in the wrong direction.

Taylor was led to his conjecture by the Pythagorean theory of 'side and diameter (diagonal) numbers', of which Plato showed his awareness by his reference in *Republic* 546c 18–19 to the rational and irrational diameters of 5. The formation of such numbers is described at length by Theon of Smyrna, ed. Hiller, 43 and by Proclus, ed. Kroll, 24–5, 27–9; and Euclid in *Elements* 2. 9, 10 gives a general geometrical proof of the formula for the generation of such numbers. In modern terminology side and diagonal numbers are pairs of numbers satisfying the equations

$$y^2 = 2x^2 \pm 1. \qquad (1)$$

Proclus gives the numbers satisfying this relationship as far as 12, 17 and Thompson extended the table as follows:

Sides (x)	Diagonals (y)
1	1
2	3
5	7
12	17
29	41
70	99
169	239

and so on.

In an isosceles right-angled triangle with unit sides the hypotenuse (diagonal of the square formed by doubling the triangle) is $\sqrt{2}$, and in the other cases the sum of the squares on the sides differs by 1 from the square on the diagonal – hence the name – the difference being alternately greater and less, thus:

$$5^2 + 5^2 = 50 = 49 + 1 = 7^2 + 1, \qquad (2)$$

$$12^2 + 12^2 = 288 = 289 - 1 = 17^2 - 1. \qquad (3)$$

Furthermore, the ratios of the diagonals to the corresponding sides,

$$\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \frac{99}{70}, \frac{239}{169}, \ldots \tag{4}$$

are increasingly close approximations to $\sqrt{2}$, alternately less and greater than $\sqrt{2}$, and in the limit the ratio becomes $\sqrt{2}$.

Taylor and Thompson saw that such a series approximating ever more closely to $\sqrt{2}$ from below and above but never terminating would satisfy the requirement of being an indefinite duality of the great and small. But $\sqrt{2}$ is only one number. Is there a general theory on these lines that would generate all numbers? Taylor saw that all surds at least could be expressed as continued fractions, and that these would satisfy the conditions.

A continued fraction is in its most general form an expression of the nature

$$a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \cfrac{b_4}{a_4 + \ldots}}} \tag{5}$$

where the a's and b's are any positive or negative integers. It may more conveniently be written

$$a_1 + \frac{b_2}{a_2+}\,\frac{b_3}{a_3+}\,\frac{b_4}{a_4+}\,\frac{b_5}{a_5+}\cdots. \tag{6}$$

In recent years mathematicians abbreviate the form still further to

$$\left(a_1, \frac{b_2}{a_2}, \frac{b_3}{a_3}, \frac{b_4}{a_4}, \frac{b_5}{a_5}, \ldots\right) \tag{7}$$

but as (6) is more recognizable by non-mathematicians we shall adhere to it here. If the continued fraction terminates, it is equivalent to a rational fraction; if it never terminates, it is a way of expressing an irrational number. By cutting the continued fraction short, thus

$$\frac{a_1}{1+}\,\frac{b_2}{a_2}, \quad \frac{a_1}{1+}\,\frac{b_2}{a_2+}\,\frac{b_3}{a_3}, \quad \text{and so on}$$

we get successively close approximations to the number represented by the continued fraction, and it can be shown that these approximations are alternately greater and less than the number. The proof is too long to give here but may be found in the several books on continued fractions such as O. Perron, *Die Lehre von den Kettenbrüche* (New York, 1950), in the articles in various encyclopaedias on continued fractions, and in O. Neugebauer, *A History of Ancient Mathematical Astronomy* III (Berlin, 1975), 1122–3.

Any surd can be expressed as a continued fraction by a simple manipulation, using the formula $(\sqrt{a}-b)(\sqrt{a}+b) = a-b^2$, thus,

$$\sqrt{a}-b = \frac{(\sqrt{a}-b)(\sqrt{a}+b)}{\sqrt{a}+b} = \frac{a-b^2}{\sqrt{a}+b} = \frac{a-b^2}{2b+(\sqrt{a}-b)}. \tag{8}$$

The formula is most readily applied when a is an integer of the form m^2+1, for in the case of $\sqrt{2}$ it reduces to

$$\sqrt{a}-1 = \frac{1}{\sqrt{a}+1} = \frac{1}{2+(\sqrt{a}-1)}. \tag{9}$$

Thus we get

$$\sqrt{2}-1 = \frac{1}{2+}\frac{1}{2+}\frac{1}{2+}\cdots \tag{10}$$

We see now that side and diameter numbers are a particular case of the convergents to a continued fraction, namely the continued fraction

$$\sqrt{2} = 1+\frac{1}{2+}\frac{1}{2+}\frac{1}{2+}\cdots. \tag{11}$$

It was therefore natural for Taylor to suspect that the explanation of the 'indefinite duality' lay in the theory of continued fractions, and although Thompson tried to damp his enthusiasm we can now see that he was right.

But when he sought confirmation of his guess he went off on the wrong track. He thought he had found it in *Epinomis* 990c5–991b4, which he interpreted as meaning that the business of geometry and stereometry was to express surd square roots and cube roots as limits of series of approximations which are alternately too large and too small. (He developed this theory further in the note on pp. 249–50 to his translation of the *Philebus* and *Epinomis*, which was published posthumously.) It is not difficult to show that surd cube roots cannot be expressed as continued fractions in the same way as surd square roots, but this might not have been realized in the Academy in Plato's time. This is, however, beside the point as Taylor read far too much into this rather playful passage, in which the author lays down a suitable education for the members of the Nocturnal Council. Its meaning is really much simpler and it is concerned mainly with the insertion in a progression of numbers of arithmetic and harmonic means (not geometric as it happens!) leading up to the music of the heavenly spheres.

The place where Taylor should have looked for confirmation was readily available but its significance was not at that time appreciated.

Euclid, *Elements* 7.1, introduces a procedure which has come to be known by mathematicians as the 'Euclidean algorithm (algorism)'.

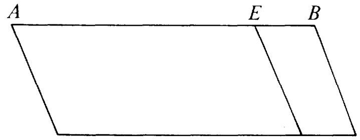
Δύο ἀριθμῶν ἀνίσων ἐκκειμένων, ἀνθυφαιρουμένου δὲ ἀεὶ τοῦ ἐλάσσονος ἀπὸ τοῦ μείζονος, ἐὰν ὁ λειπόμενος μηδέποτε καταμετρῇ τὸν πρὸ ἑαυτοῦ, ἕως οὗ λειφθῇ μονάς, οἱ ἐξ ἀρχῆς ἀριθμοὶ πρῶτοι πρὸς ἀλλήλους ἔσονται.

'Two unequal numbers being set out, and the lesser being continually subtracted in turn from the greater, if the remainder never measures the number preceding it until a unit is left, the original numbers will be prime to one another.'

In 7. 2 Euclid uses this procedure to find the greatest common measure of two numbers not prime to one another. In 10. 2 he shows that if this procedure is applied to two magnitudes of the same kind and never comes to an end the magnitudes are incommensurable, and in 10. 3 he uses it to find the greatest common measure of two magnitudes which are commensurable. In all four passages the genitive singular of the present participle passive, *ἀνθυφαιρουμένου*, of the verb *ἀνθυφαιρεῖν*, 'to take away again', 'to take away in turn', is used. For this reason the procedure could be called 'continuing (or alternating or reciprocal) subtraction', but in default of a single English word it is generally known by a transliteration of the Greek noun as *anthyphairesis*.

These propositions did not attract much attention until in 1933 Oskar Becker drew attention to a relevant passage in Aristotle's *Topics* and the comment of Alexander Aphrodisiensis thereon. In English they are:

'It would seem that in mathematics also some things are not easily proved for lack of a definition, such as the proposition that the straight line parallel to the side which cuts the plane (i.e. a parallelogram) divides in the same way both the line and the area. But when the definition is stated, what was said becomes immediately clear. For the areas and the lines have the same *antanairesis* (ἀνταναίρεσις); and this is the definition of the same proportion.' (Aristotle, *Topics* 8. 3, 158b29–35.)



Clearly what Aristotle means is that in a parallelogram *ABCD* if *EF* is drawn parallel to the sides *AD*, *BC* so as to cut *AB* in *E* and *DC* in *F*, then *EF* divides the parallelogram in the same ratio that it divides the lines *AB*, *DC*.

'For likewise when this is stated it is not obvious; but when the definition of proportion is enunciated it becomes obvious that both the line and the area are cut in the same proportion by the line drawn parallel. For the definition of proportions which those of old time used is this: Magnitudes which have the same *anthyphairesis* are proportional. But *anthyphairesis* he has called *antanairesis*.' (Alexander Aphrodisiensis, *On the Topics*, 8. 3, ed. Wallies 595. 12–17.)

A. W. Pickard-Cambridge's translation of the passage in the *Topics* for the Oxford translation of Aristotle is, 'the areas have the same fraction subtracted from them as have the sides; and this is the definition of "the same ratio"'. This had been the normal translation. It would accord with the Pythogorean definition of proportional numbers given by Euclid in *Elements* 7, Def. 20, applicable only to commensurable magnitudes. But in 1933 Oskar Becker drew attention to the significance of Alexander's comment and to the apparent use by Euclid of ἀνθυφαιρεῖν as a technical term for finding the greatest common measure of two quantities and testing their commensurability with each other. If the process of *anthyphairesis* is applied to the terms of a ratio it yields a series of quotients, and if the application of the procedure to the terms of a second ratio yields the same series of quotients the two ratios must be equal: and it would therefore be reasonable for 'those of old time' to say that two ratios are equal if they have the same *anthyphairesis*. Such a definition applies equally to magnitudes and to numbers, and to incommensurable as well as commensurable magnitudes or numbers, and this led Becker to conjecture that there was a general theory of proportion before that discovered by Eudoxus and incorporated in Euclid, *Elements* 5; and he attempted to reconstruct it in a formal series of propositions (*Quellen und Studien*, B2, 1933, 311–33, 369–87; B3, 1936, 236–49, 370–88, 389–410).

It took some time for Becker's theory to be appreciated, especially as the Second World War broke out in 1939, and I do not know whether Taylor had become aware of it before he died in 1945. In his *Mathematics in Aristotle*, 81–3, posthumously published in 1949, T. L. Heath showed his awareness, but could 'only regard Becker's article as a highly interesting speculation'. K. Reidemeister would have limited the theory to commensurables, and Dr Árpád Szabó has been sceptical. But in general students of Greek mathematics gradually became convinced of the soundness of Becker's interpretation and as they pursued the matter they realized that there were traces of the anthyphaeretic theory in the Arabic commentators on Euclid, and that to some extent Becker had been anticipated by H. G. Zeuthen and E. J. Dijksterhuis.

No one has done more to emphasize the importance of anthyphairesis in the development of Greek mathematics than Mr D. H. Fowler, approaching the subject from the standpoint of a practising mathematician at the University of Warwick, and in one of his many papers on the subject he attributes the invention of the theory to Theaetetus. (Becker had ascribed the general definition of incommensurability by means of *antanairesis* to Theodorus and held that in the time of Theaetetus it was applied to numbers.)

If it is the case that the Greeks had a general theory of proportion (that is, a theory applicable to incommensurable as well as commensurable magnitudes and numbers) based on *anthyphairesis*, there is nothing surprising in the fact that virtually all trace of it was lost when it was superseded by the still more powerful theory of Eudoxus embodied in Euclid, *Elements* 5, a theory so satisfying that it has remained acceptable to this day. There is a good parallel. We should never have known that Euclid's *Elements* were preceded by earlier *Elements of Geometry* written by Hippocrates of Chios, Leon and Theudius of Magnesia if the fact had not been mentioned by Proclus. Not a single line of these earlier works has been preserved: they were swept out of existence by the superior merits of Euclid's composition.

It has long been recognized that the subject of incommensurability was one of the major interests of the mathematical department of the Academy, which included in Theaetetus and Eudoxus two of the greatest mathematicians of all time, and it would now appear that in the Academy there were successively developed two general theories of proportion – the anthyphairetic, whether invented by Theaetetus or not, and the multiple theory due to Eudoxus. Though Plato was not a practising mathematician, his dialogues show that he took the greatest interest in it and gave every encouragement to its study. He could hardly fail to be interested in the fascinating general theories that his gifted pupils were developing, more particularly as they exemplified his own ideal of finding a unity in multiplicity.

If it is the case that the anthyphairetic theory was developed in his own class rooms, it becomes easy to see how Plato conceived that number was generated by a duality of the great and small. For the anthyphairetic process is exactly equivalent to the modern theory of continued fractions. Neat proofs are given by Dr O. Neugebauer in *A History of Ancient Mathematical Astronomy* III, 1120–1 and by Mr D. H. Fowler, *Bulletin of the American Mathematical Society*, 'Ratio in Early Greek Mathematics', I (1979), 841–5, If $a > b > 0$ and a and b are relatively prime (i.e. their greatest common divisor is 1), the Euclidean algorithm or the process of *anthyphairesis* is the following sequence

$$a = q_0 b + r_1, \quad b = q_1 r_1 + r_2, \quad r_1 = q_2 r_2 + r_3,$$

and so on, the process terminating after a finite number of steps if a/b is commensurable and not terminating if it is incommensurable. For convenience write

$$\frac{a}{b} = x_0, \quad \frac{b}{r_1} = x_1, \quad \frac{r_1}{r_2} = x_2 \quad \text{and so on.}$$

The algorithm now becomes

$$x_0 = q_0 + \frac{1}{x_1}, \quad x_1 = q_1 + \frac{1}{x_2}, \quad \text{and so on,}$$

so that

$$\frac{a}{b} = q_0 + \cfrac{1}{q_1 + \cfrac{1}{q_2}} \cdots$$

or
$$\frac{a}{b} = q_0 + \frac{1}{q_1+}\,\frac{1}{q_2+}\cdots.$$

As already stated, it can be shown that the successive convergents of this continued fraction are alternately less and greater than a/b.

It would, of course, be absurd to maintain that the mathematicians of the Academy were familiar with the theory of continued fractions as discovered by Euler and Lagrange. Indeed, they had no general notation for fractions, much less for continued fractions. But the mental processes in the Euclidean algorithm, *anthyphairesis* and continued fractions are the same, and there is confirmatory evidence, as we shall see, that the ancient Greek mathematicians achieved the same results by the Euclidean algorithm or *anthyphairesis* that modern mathematicians achieve by continued fractions. But before it is given, two points may be made. The first is that there is a parallel. Archimedes performed evaluations of areas and volumes by methods that are equivalent to the integral calculus though he had no notation comparable with that of Newton or Leibniz when they developed the modern science. The second is that the Euclidean algorithm is so called because we know it from Euclid's *Elements*; but that work is a compilation embodying in the main much older discoveries. There is no reason to think that the algorithm was not discovered until about 300 B.C.; it was probably fully in use in the Academy in Plato's time, and perhaps earlier.

It was said above that there is confirmatory evidence that the Greeks performed operations equivalent to continued fractions. In the *Measurement of a circle* Archimedes baldly states (in effect)

$$\tfrac{1351}{780} > \sqrt{3} > \tfrac{265}{153},$$

and the method by which he reached these approximations puzzled historians until it was realized that the fractions are convergents to continued fractions. The simplest way of expanding the square root of 3 as a continued fraction is

$$\sqrt{3} = 1 + \frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{2+}\cdots \tag{12}$$

and the convergents are

$$\tfrac{1}{1}, \tfrac{2}{1}, \tfrac{5}{3}, \tfrac{7}{4}, \tfrac{19}{11}, \tfrac{26}{15}, \tfrac{71}{41}, \tfrac{97}{56}, \tfrac{265}{153}, \tfrac{362}{209}, \tfrac{989}{571}, \tfrac{1351}{780}, \ldots. \tag{13}$$

It will be seen that the fractions used by Archimedes are the ninth and twelfth in this series. This raises the question why Archimedes did not use the eleventh convergent, $\tfrac{989}{571}$, for his under-estimate. There is, however, an expansion of $\sqrt{3}$ as a continued fraction in which Archimedes' figures do appear as successive convergents. Thompson, prompted by that eminent mathematician H. W. Turnbull, and Mr Fowler, in a slightly different way in the paper cited, show that

$$3\sqrt{3} = 5 + \frac{1}{5+}\,\frac{1}{10+}\,\frac{1}{5+}\,\frac{1}{10+}\cdots \tag{14}$$

to which the convergents are $\tfrac{5}{1}, \tfrac{26}{5}, \tfrac{265}{51}, \tfrac{1351}{260}, \ldots$ giving the approximations to $\sqrt{3}$ as $\tfrac{5}{3}, \tfrac{26}{15}, \tfrac{265}{153}, \tfrac{1351}{780}$. The fact that Archimedes' approximations are successive convergents in this series and are reached after only three and four steps (instead of nine and twelve) make it plausible that this was the expansion used by Archimedes.

A little earlier Aristarchus in his work on *The Sizes and Distances of the Sun and Moon* had stated baldly that $\tfrac{7921}{4050} > \tfrac{88}{45}$ and $\tfrac{71\,755\,875}{61\,735\,500} > \tfrac{43}{37}$. If $\tfrac{7921}{4050}$ is developed as a continued fraction we get the approximation $1 + \tfrac{1}{1+}\,\tfrac{1}{21+}\,\tfrac{1}{2}$, which is $\tfrac{88}{45}$, and if $\tfrac{71\,755\,875}{61\,735\,500} = \tfrac{21\,261}{18\,292}$ is similarly expanded we get the approximation $1 + \tfrac{1}{6+}\,\tfrac{1}{6}$, which is $\tfrac{43}{37}$.

It has been pointed out to me that Archimedes could simply have proved his inequality in the following manner:

$$\left(\frac{1351}{780}\right)^2 = \frac{1825201}{608400} = \frac{3\times 608400+1}{608400} > 3,$$

while

$$\left(\frac{265}{153}\right)^2 = \frac{70225}{23409} = \frac{3\times 23409-2}{23409} < 3.$$

Likewise it has been pointed out to me that Aristarchus could simply have shown that

$$7921\times 45 = 356445 > 356400 = 4050\times 88.$$

This is true, but what calls for explanation is why Archimedes started with such curious fractions as $\frac{1351}{780}$ and $\frac{265}{153}$, and Aristarchus with such fractions as $\frac{7921}{4050}$ and $\frac{88}{45}$. It is at least worth investigation that they occur as convergents in continued fractions even though other explanations may also be possible.

To avoid misunderstanding let it again be repeated that there is no suggestion here that Archimedes or Aristarchus or any other ancient Greek was familiar with the theory of continued fractions. They would have achieved their results by the Euclidean algorithm or by anthyphairesis. But the results and the mental processes are the same. For a modern mathematician convergent fractions are a convenient way of handling such questions, but they can, though sometimes not without difficulty, be handled anthyphairetically, and in a paper to be published shortly with the provisional title 'Eratosthenes' ratio for the obliquity of the ecliptic' Mr D. H. Fowler does, in fact, give a completely anthyphairetic proof of Archimedes' inequality in which $\frac{265}{153}$ and $\frac{1351}{780}$ are the third and fourth convergents.

Both Aristarchus and Archimedes lived after Plato – perhaps a hundred years after his *floruit* – but the fact that they state these inequalities without proof or explanation shows that the approximations were in use before their time, and may very well have been bandied about in the Academy while Plato was still at its head. There is one piece of evidence which may show that a process equivalent to continued fractions was known before the foundation of the Academy. In a much-discussed passage, *Theaetetus* 147d–148b, Plato makes Theaetetus describe how Theodorus demonstrated to him and the younger Socrates that the square roots of 3, 5 and so on up to 17 are incommensurable with the unit, but he stopped with 17 because 'somehow he got into difficulties' (*πως ἐνέσχετο*). There are several plausible explanations why Theodorus stopped at this point, but the most likely is that he used the Euclidean algorithm to test the commensurability of the various numbers. The square root of 17 can easily be shown to be incommensurable because it is of the form $\sqrt{4^2+1}$ and

$$\sqrt{17}-4 = \frac{(\sqrt{17}-4)(\sqrt{17}+4)}{\sqrt{17}+4}$$

$$= \frac{1}{8+(\sqrt{17}-4)}$$

so that

$$\sqrt{17} = 4+\frac{1}{8+}\,\frac{1}{8+}\,\frac{1}{8+} \qquad (15)$$

As $\sqrt{18} = 3\sqrt{2}$ and $\sqrt{2}$ is known to be incommensurable it does not call for discussion but the evaluation of $\sqrt{19}$ as a continued fraction presents formidable difficulties as recurrence does not occur till after six stages. The stages of applying the Euclidean algorithm to $\sqrt{19}$ are set out by Professor B. L. van der Waerden algebraically in *Science awakening* (Groningen, 1961) 145, and by myself arithmetically in the *Dictionary of Scientific Biography*, 'Theodorus of Cyrene', vol. XIII, 317, and the expansion of $\sqrt{19}$ as a continued fraction, given by H. Davenport, *The Higher Arithmetic* (London, 1952), 104, and by D. H. Fowler, *Sides and Diameters, Continued Fractions and Pell's Equation*, 22, is

$$\sqrt{19} = 4 + \frac{1}{2+}\frac{1}{1+}\frac{1}{3+}\frac{1}{1+}\frac{1}{2+}\frac{1}{8+}\frac{1}{2+}\frac{1}{1+}\frac{1}{3+}\frac{1}{1+}\frac{1}{2+}\frac{1}{8+}\cdots \tag{16}$$

This confirms the remarkable property that the expansion of the square root of an integer which is not a perfect square as a continued fraction is periodic, its period starting with the second term, that the period is terminated by a term twice the first term, and that the period up to the final term is palindromic.

Though such sophisticated properties would hardly have been thought of in the Academy, the difficulties of applying the Euclidean algorithm to $\sqrt{19}$ would have been readily appreciated and make it easy to understand why Theodorus stopped at $\sqrt{17}$. He would have found sufficient difficulties with $\sqrt{13}$ to warn him! If this interpretation is correct – and it must be regarded only as the most likely of several – it is evidence that the procedure of successively closer approximations to a surd from above and below was known even before Plato founded the Academy.

Finding the greatest common measure, the Euclidean algorithm, anthyphairesis, continued fractions – we see that these are different names for the same mental process. They give successively closer approximations to a number from above and below and justify the belief that Plato was referring to it in the phrase 'the duality of the great and small'.

But it is also an 'indefinite' duality, and this point must now be considered. In the case of a surd, the sequence of convergents is endless, and the duality is therefore justly called 'indefinite' or 'unbounded'. But what of the rational numbers? An integer does not need to be expressed as a continued fraction, and the ratio of two integers expanded as a continued fraction terminates after a finite number of steps. Prima facie, integers and rational fractions are not expressed by an indefinite duality. Did the Academicians merely ignore this problem or sweep it under their desks? With their reputation for rigour it is more likely that they realized that an integer or a rational fraction *can* formally be expressed by an infinite number of terms, thus

$$2 = 2 + \frac{0}{1+}\frac{0}{1+}\frac{0}{1+}\cdots \tag{17}$$

and

$$\frac{3}{2} = 1 + \frac{1}{2+}\frac{0}{1+}\frac{0}{1+}\cdots \tag{18}$$

They would, however, have done it in their own way by continuing the divisions in the Euclidean algorithm even after a unit is left as the remainder in the case of prime numbers, or the greatest common measure has been found in the case of numbers not prime to each other.

So much for the indefinite duality of the great and small. But what of the 'one' which Plato says is 'mixed' with it to produce number?

There has been unnecessary confusion over the meaning of this passage by giving 'the one' a metaphysical meaning, honouring it with a capital letter in translation ('the One'), and effectively identifying it with the Absolute of the Idealist philosophers. This interpretation has been assisted by the intrusion into the text of *Met.* 987b 22 quoted above of the words τὰ εἴδη before εἶναι τοὺς ἀριθμούς. These words make it impossible to translate the sentence satisfactorily, and Zeller rightly saw that they must be excluded. Clearly a metaphysical meaning should not be sought unless a purely mathematical rendering proves impossible to find, for the passage is ostensibly mathematical.

Thompson thought he had found a simple mathematical meaning in the fact that double the square on the side number in the case of the side and diagonal numbers giving approximations to $\sqrt{2}$ is always 1 greater or 1 less than the square on the diagonal number. The general formula, it will be recalled, is

$$2x^2 = y^2 \pm 1$$

and the first four examples are

$$2\times 1^2 = 1^2+1, \quad 2\times 2^2 = 3^2-1, \quad 2\times 5^2 = 7^2+1, \quad 2\times 12^2 = 17^2-1.$$

Some plausibility is lent to this theory by a passage in Aristotle's *Metaphysics* (1081a24–5), εἴτε ὥσπερ ὁ πρῶτος εἰπὼν (sc. Plato) ἐξ ἀνίσων (ἰσασθέντων γὰρ ἐγένοντο) εἴτε ἄλλως, which implies that the function of the one (τὸ ἕν) is to 'equalize' or 'make equal' (ἰσάζειν) the unequal. But Thompson saw himself immediately that this simple explanation would not suffice. It is derived from the side and diagonal numbers for $\sqrt{2}$, but the corresponding table for $\sqrt{3}$ is

$$3\times 1^2 = 2^2-1, \quad 3\times 3^2 = 5^2+2, \quad 3\times 4^2 = 7^2-1, \quad 3\times 11^2 = 19^2+2,$$

that is, 'the "One" is no longer the unique and indispensable "equalizer"'; and he shows that 'it is by no means indispensable (though at first it seemed to be so) in the series of side and diagonal numbers which leads to $\sqrt{2}$'. Indeed, he extends Theon's table to show that the 'equalizing factor' measuring the amount of excess or defect may be ± 1, ± 2, ± 4, ± 8 and so on in one direction or $\pm\frac{1}{2}$, $\pm\frac{1}{4}$ and so on in the other direction (loc. cit. 48–50).

What then is the true function of the one, the unit, the monad (τὸ ἕν)? I believe it has escaped understanding only because it is so simple. Interpreters have been looking for something more complex and sophisticated than Plato intended.

It has been debated whether the discovery of the irrational was a 'scandal' or not, but certainly the discovery that there were numbers and magnitudes which were not commensurable with the unit of number or magnitude must have been traumatic for the early Greek mathematicians at whatever point the discovery was made. The discovery of transcendental numbers in modern times may give us some inkling of the pleasure or pain that the discovery of the irrational gave to the Greeks, but the shock of that early discovery must have been immensely greater because it was the first such discovery and modern mathematicians have become accustomed to extending their definitions to include hitherto unknown properties. We know that immense attention was paid to the problem.

To the mathematicians of the Academy, in particular, with their faith in the human reason and their desire to subsume many particulars under one general rule or principle – this is what the doctrine of Forms or Ideas is about – it must have been a consuming ambition to find a treatment that would bring rational and irrational numbers and magnitudes together in one theory. This is what they achieved in their doctrine that number is formed by the operation of the one (as formal element) upon the indefinite duality of the great and small (as the material element). Certain numbers and magnitudes, it had been discovered, could not be expressed in terms of integers;

yes they could, the mathematicians of the Academy triumphantly proclaimed, *if there are enough of them*. We may legitimately compare the discovery by Wallis (*Arithmetica infinitorum*, 1665) that π could be expressed in terms of an infinite product of rational fractions –

$$\frac{4}{\pi}=\frac{3}{2}\cdot\frac{3}{4}\cdot\frac{5}{4}\cdot\frac{5}{6}\cdot\frac{7}{6}\cdots, \tag{19}$$

or, even more relevantly to our present purpose, the proof by his friend Lord Brouncker that π can be expressed in terms of an infinite continued fraction containing only integers –

$$\frac{4}{\pi}=1+\frac{1^2}{2+}\frac{3^2}{2+}\frac{5^2}{2+}\frac{7^2}{2+}\cdots. \tag{20}$$

And what is an integer but a unit or a collection of units? The very first definition of number in the history of Greek mathematics – by Thales according to Iamblichus – was μονάδων σύστημα, 'a collection of units', and Plato's brightest pupil, Eudoxus, defined number as πλῆθος ὡρισμένον, 'a determinate multitude'. In yet another relevant definition Nicomachus says that number is ποσότητος χύμα ἐκ μονάδων συγκείμενον, 'a flow of quantity composed of units'. Whatever metaphysical views Plato himself may have entertained about the nature of mathematical objects, this is the view of whole numbers that would have been held by the practical mathematicians of the Academy. In all probability it is they who, by the application of the Euclidean algorithm, discovered that surds, hitherto 'inexpressible' (ἄρρητα or ἄλογα), can be expressed by means of two infinite series of rational fractions converging on the limit, one from above and one from below. This is something that was well within the capacity of such men as Theaetetus and Eudoxus, and although Plato would not himself have been able to discover the method he may very well (as Aristotle implies) have been responsible for the formulation that number comes about by the operation of the one upon the indefinite duality. Though the procedure was discovered in relation to surds, it would have been a mere formal extension, as we have seen, to apply it to integers and rational fractions; and thus a general theory of number was formulated, bringing commensurables and incommensurables under one definition.

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